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Chapter 4

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A Model of Technology and Heterogeneous Costs

Here we develop a model of technology and costs that underlies both our static general equilibrium analysis of trade flows and our productivity dynamics. The chapter is divided into four sections. The first sets out our basic assumption about ideas that underlies the remainder of the book. The second two provide a technical derivation of the properties of the distribution of costs implied by our assumption. The final section summarizes what we use of these properties in the ensuing chapters. The reader not interested in the probability theory behind the results should read the next section but can then safely skip to the last one, where we summarize what's relevant for the remainder of the book.

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4.1 Ideas, Techniques, and Unit Costs

The fundamental atom of technology is an idea. An idea is a recipe to produce some good j with some efficiency q (which we call the quality of the idea). Efficiency is simply the amount of output that can be produced with a unit of input. In this formulation, both output and input are measured in units of constant quality.

At any moment, a location is characterized by the ideas available to it for production, and an input cost w_i. (In Chapter 6, where we introduce trade among locations, their geography relative to one another becomes another important feature.) In this chapter we take w_i as given, and derive how the stock of ideas available at a location at any moment are determined by the history of their arrival.

Connecting an idea (for making a good j with efficiency q) with location i gives rise to a technique for producing the good there at unit cost w_i/q. For now we focus on ideas about a particular good j at a single location i. We will then make assumptions about the range of goods, which could be exogenous or endogenous, constant or growing over time. We defer multiple locations to Chapter 6.

Time is continuous. Ideas for good j arrive at location i at date t according to a Poisson process with intensity $\overline{a}R_i(j,t)$. We can think of R as reflecting research effort and $\overline{a}$ (to be normalized shortly) as reflecting research productivity. The quality of an idea is the realization of a random variable Q drawn independently from the Pareto distribution with parameter $\theta > 1$, so that:

$$\begin{aligned} \Pr[Q &> q] = \left(q/\underline{q}\right)^{-\theta} \quad q \geq \underline{q} \\ &= 1 \quad q < \underline{q} \end{aligned} \tag{4.1}$$

where $\underline{q} > 0$ is the minimum quality level. A useful property of the Pareto distribution is that, conditional on an idea being better than q', the probability that the idea is better than q, for any $q \geq q'$, is:

$$\Pr[Q > q | Q \geq q'] = (q/q')^{-\theta}. \tag{4.2}$$

That is, given that the idea is better than q', the probability distribution of its quality is Pareto with parameter θ and lower bound q'.

Together these assumptions imply that for any $q > 0$, the arrival rate of ideas of efficiency $Q \geq q$ is

$$\overline{a} R_i(j,t) \left(q/\underline{q}\right)^{-\theta}.$$

In this formulation there is no inherent distinction between $\overline{a}$ and the minimum quality of an idea $\underline{q}$. Hence we normalize $\overline{a}\underline{q}^{\theta} = 1$ so that the arrival rate of ideas of efficiency greater than q simplifies to $R_i(j,t)q^{-\theta}$. Taking the limit as $\underline{q} \to 0$ (and, hence, $\overline{a} \to \infty$ so that $\overline{a}\underline{q}^{\theta}$ remains at unity) allows us to consider ideas of all qualities in the domain $(0, \infty)$. In what follows $\overline{a}$ and $\underline{q}$ can be ignored.

We assume that there is no forgetting; that is, once an idea has arrived at a location it is available for production thereafter. The number of ideas available for producing good j thus reflects the history of the arrival rate of ideas about good j at location i by date t. We summarize this history with the term $T_i(j,t)$ given by the integral:

$$T_i(j,t) = \int_{-\infty}^{t} R_i(j,\tau)d\tau.$$

Our assumptions imply that the number of ideas about good j with quality $Q > q'$ is distributed Poisson with parameter $T_i(j,t)\,(q')^{-\theta}$. The distribution of quality among these ideas is given by (4.2).

Since a bundle of inputs at location i costs w_i, the unit cost of producing good j at that location with a technique of efficiency q is $c = w_i/q$. This unit cost is itself the realization of a random variable C.

We now turn to the distribution of the random variable C. The key parameter for this distribution is:

$$\Phi_i(j,t) = T_i(j,t)w_i^{-\theta}. \tag{4.3}$$

which combines the history of the arrival of ideas together with input costs. The following proposition characterizing properties of the set of techniques with unit cost $C \leq c$ is immediate:

Proposition 1 *Given $\Phi_i(j,t)$: (i) The number of techniques providing unit cost less than c is distributed Poisson with parameter $\Phi_i(j,t)c^{\theta}$. (ii) The expected number of such techniques is $\Phi_i(j,t)c^{\theta}$. (iii) The conditional distribution of unit costs using these* *techniques is:*

$$\Pr[C \leq c'|C \leq c] = \Pr\left[Q \geq \frac{w}{c'}|Q \geq \frac{w}{c}\right] = (c'/c)^{\theta} \quad c' \leq c, \tag{4.4}$$

which is invariant to input costs w and the technology parameter T.

As ideas arrive at a location over time, there will be many available recipes for producing good j. At any time t we can rank the techniques available at location i according to their implied unit costs $C^{(1)} \leq C^{(2)} \leq C^{(3)} \leq \ldots$. For the time being our analysis does not depend on location i, time t, or good j, so we suppress these arguments, reintroducing them when they become relevant. Thus, in what follows next, we simply set $\Phi = \Phi_i(j,t)$.

The next two sections present some basic properties of the distribution of the order statistics $C^{(k)}$, $k = 1, 2, 3, ...$, for given Φ. Section 4.4 summarizes what is needed of these results for the subsequent analysis. The reader not interested in the probability theory behind them can skip ahead.

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4.2 The Basic Theorem

Our ensuing analysis is based on the many layers of costs for a good, starting with the lowest unit cost $C^{(1)}$, then the second-lowest $C^{(2)}$, and working up from there (with the economics of any particular application telling us how many layers we need to go up). The following theorem characterizes the joint distribution of these layers of unit costs for a particular good, where $C^{(k)}$ denotes the k'th lowest unit cost technology for producing it. This theorem on the distribution of costs serves as the basis for many of our subsequent results.

Theorem 1 *The joint density of $C^{(k)}$ and $C^{(k+1)}$ is:*

$$g_{k,k+1}(c_k, c_{k+1}) = \frac{\theta^2}{(k-1)!}\Phi^{k+1} c_k^{\theta k-1} c_{k+1}^{\theta-1} \exp[-\Phi c_{k+1}^{\theta}]$$

for $0 < c_k \leq c_{k+1} < \infty$ while the marginal density of $C^{(k)}$ is:

$$g_k(c_k) = \frac{\theta}{(k-1)!}\Phi^k c_k^{\theta k-1} \exp[-\Phi c_k^{\theta}].$$

for $0 < c_k < \infty$.

The Appendix to this Chapter presents the proof of the theorem and the lemmata below.

Theorem 1 characterizes the joint distribution of each pair of adjacent order statistics. By induction these distributions are sufficient to characterize the full distribution across any number of ordered unit costs, i.e., $C^{(1)}, C^{(2)}, C^{(3)}, \ldots, C^{(k)}$ for any finite integer k. (See Karlin and Taylor, Chapter 13, 1981.) Note that the distributions depend only on the two parameters θ and Φ $(= Tw^{-\theta})$. Hence the parameter $\Phi = \Phi_i(j,t)$ summarizes all we need need to know about time t, location i, and good j for the distribution of costs. The theorem thus provides a connection between $T_i(j,t)$, the parameter governing the history of ideas that have arrived at a location i about good j by time t, and input costs $w_i(t)$ there at time t, to the distribution of the costs of making it.

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4.3 Probabilistic Implications

With this central result in hand we are able to show a number of features about the cost distribution that we apply repeatedly in the following chapters. The first two lemmata give the distribution of the k'th lowest cost and its moments.

Lemma 1 *The distribution of the k'th lowest cost* $C^{(k)}$ *is:*

$$\Pr[C^{(k)} \leq c_k] = F_k(c_k) = 1 - \sum_{i=0}^{k-1} \frac{\left(\Phi c_k^{\theta}\right)^i}{i!} \exp\left[-\Phi c_k^{\theta}\right],$$

Of particular interest for what follows is the distribution of the lowest cost $C^{(1)}$. Setting $k = 1$ gives the (inverse) Type 2 extreme value (or Fréchet) distribution:

$$F_1(c_1) = \exp(-\Phi c_1^{\theta}).$$

In our applications below, whether we need to probe into further layers depends on our assumptions about market structure and the ownership of technology.

Say that a large number of potential producers have access to the lowest-cost technology and compete perfectly with each other to produce good j at cost $C^{(1)}$. In this case only the distribution of $C^{(1)}$ is of interest, since it applies to both cost and price.

Say, instead, that only a single producer has access to the lowest cost technology (due, for example, to patent protection or trade secrecy), while at least one other producer has access to the second-lowest cost technology. Under Bertrand competition the cost distribution is also given by the frontier ($k = 1$) but prices are related to the distribution of the second lowest cost ($k = 2$).

Say that each technology is available to only a single potential producer, and these potential producers are Cournot competitors. Then higher values of k could be relevantant. The following Chapter explores different forms of competition in greater depth.

The second lemma is useful in calculating price indices:

Lemma 2 *For each order k, the b'th moment ($b > -\theta k$) is:*

$$E[(C^{(k)})^b] = \left(\Phi^{-1/\theta}\right)^b \frac{\Gamma\left[(\theta k + b)/\theta\right]}{(k-1)!}$$

where $\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y} dy$ is the Gamma function.

This lemma provides a link between the state of technology and wages, as reflected in Φ, and moments of costs at various tiers k. The homogeneity of prices with respect to costs then implies a link between technology and wages, on one hand, and the price index, on the other.[1]

[1] In the following chapters we assume a CES aggregator across goods, with elasticity of substitution σ. Setting $b = 1 - \sigma$ in Lemma 2 implies that the price index is homogeneous of degree 1 in the wage

We have now characterized the various layers of the cost distribution. We will also be using results on the of the distribution of one layer conditional on the realization of an adjacent one. The next two lemmata concern the distribution of the $k+1$'st lowest unit cost given the realization of the k'th.

Lemma 3 *The distribution of $C^{(k+1)}$ conditional on $C^{(k)} = c_k$ is:*

$$\Pr[C^{(k+1)} \leq c_{k+1} | C^{(k)} = c_k] = 1 - \exp\left[-\Phi(c_{k+1}^{\theta} - c_k^{\theta})\right] \qquad c_{k+1} \geq c_k \geq 0$$

Lemma 4 *The distribution of the ratio of $C^{(k+1)}$ to $C^{(k)}$ conditional on $C^{(k)} = c_k$ is:*

$$\Pr\left[\frac{C^{(k+1)}}{C^{(k)}} \leq m | C^{(k)} = c_k\right] = 1 - \exp\left[-\Phi c_k^{\theta}(m^{\theta} - 1)\right].$$

Reversing the conditioning order of the previous two lemmata, the next two concern the distribution of the k'th layer given the realization of the $k+1$'st.

Lemma 5 *The distribution of $C^{(k)}$ conditional on $C^{(k+1)} = c_{k+1}$ is:*

$$\Pr[C^{(k)} \leq c_k | C^{(k+1)} = c_{k+1}] = \left(\frac{c_k}{c_{k+1}}\right)^{\theta k} \qquad c_{k+1} \geq c_k \geq 0.$$

Lemma 6 *The distribution of the ratio of $C^{(k+1)}$ to $C^{(k)}$ conditional on $C^{(k+1)} = c_{k+1}$ is:*

$$\Pr\left[\frac{C^{(k+1)}}{C^{(k)}} \leq m | C^{(k+1)} = c_{k+1}\right] = 1 - m^{-\theta k}.$$

(Hence, $C^{(k+1)}/C^{(k)}$ is independent of $C^{(k+1)}$.)

and of degree $-1/\theta$ in the state of technology T. That is, given the wage an increase in T lowers the price index with an elasticity $-1/\theta$.

This result will prove useful in describing the distribution of the markup of price over cost under Bertrand competition.

For some purposes it is convenient to work with transformed costs defined as $U^{(k)} = \Phi\left(C^{(k)}\right)^{\theta}$ for $k = 1, 2, 3, ...$The joint distribution of the $U^{(k)}$ has a very simple structure:

>From Lemma 1, for $k = 1$:

$$\begin{aligned} \Pr[U^{(1)} &\leq u_1] = \Pr[\Phi\left(C^{(1)}\right)^{\theta} \leq u_1] = \Pr\left[C^{(1)} \leq \left(\frac{u_1}{\Phi}\right)^{1/\theta}\right] \\ &= 1 - \exp(-u_1) \end{aligned} \tag{4.5}$$

while from Lemma 3:

$$\begin{aligned} \Pr[U^{(k+1)} &\leq u_{k+1}|U^{(k)} = u_k] = \Pr\left[\Phi\left(C^{(k+1)}\right)^{\theta} \leq u_{k+1}|\Phi\left(C^{(k)}\right)^{\theta} = u_k\right] \\ &= \Pr\left[C^{(k+1)} \leq \left(\frac{u_{k+1}}{\Phi}\right)^{1/\theta}|C^{(k)} = \left(\frac{u_k}{\Phi}\right)^{1/\theta}\right] \\ &= 1 - \exp\left[-(u_{k+1} - u_k)\right]. \end{aligned} \tag{4.6}$$

This result allows us to draw a series of transformed costs, starting with the lowest cost and working up, from the unit exponential distribution (hence, with no parameter values needed). The costs themselves can then be recovered by applying the inverse transformation, $C^{(k)} = \left(U^{(k)}/\Phi\right)^{1/\theta}$, which depends on the two parameters, Φ and θ. The process is analogous to building up a general multivariate normal distribution from independent standard normal distributions. This technique is directly applicable in simulation where it is advantageous to isolate the parameters of the model from the stochastic elements of the model.

This result also leads to the following lemma about any function $H(C^{(1)}, C^{(2)}, C^{(3)}, ...)$ homogenous of degree one in ordered unit costs costs.[2]

Lemma 7 *A function $H(C^{(1)}, C^{(2)}, C^{(3)}, ...)$ that is homogeneous of degree one in ordered unit costs can be written*

$$H(C^{(1)}, C^{(2)}, C^{(3)}, ...) = \Phi^{-1/\theta} H\left[\left(U^{(1)}\right)^{1/\theta}, \left(U^{(2)}\right)^{1/\theta}, \left(U^{(3)}\right)^{1/\theta}, ...\right]$$

where the joint distribution of the $U^{(k)}$'s are given by (4.5) and (4.6) above.

The result follows immediately from the relationship $C^{(k)} = \Phi^{-1/\theta}\left(U^{(k)}\right)^{1/\theta}$. Since $\Phi^{-1/\theta} = T^{-1/\theta}w$, any linear homogeneous function of unit costs is proportional to the cost of an input bundle. We use this result to derive general properties of the price index in the next chapter.

4.4 Aggregate Implications

So far we have dealt with the unit cost of producing some particular good j. We now integrate these results into a model of the aggregate economy. As in the Ricardian model with a continuum of goods and the quality ladders model (discussed in Chapter

[2]The function need not actually depend on all the ordered costs. For example, the function $H(C^{(1)}, C^{(2)}, C^{(3)}, ...) = \alpha C^{(1)}$ for $\alpha > 0$ has the required linear homogeneity property.

3) we assume that there are a measure J of goods such that $j \in [0, J]$.[3] Also following this literature (and monopolistic competition as well), we assume that inputs are mobile across the production of these goods in some country i, so producing any of them entails paying the same input cost w_i (since we assume that production of all goods uses the same combination of inputs).

As in the quality ladders literature, we treat overall research activity as not directed to a specific good j. What good an idea is relevant for is drawn uniformly across $[0, J]$. As a result, the number of ideas that arrive, as well their quality, are independent across goods.

We specify the aggregate flow of ideas into location i at date τ with quality better than q as $R_i(\tau)q^{-\theta}$. Since these ideas fall randomly across the continuum, the number applicable to good j is distributed Poisson with parameter $R_i(\tau)q^{-\theta}/J$.[4]

We can summarize the history of the arrival of ideas to location i by time t with the term:

$$T_i(t) = \int_{-\infty}^{t} R_i(\tau)d\tau.$$

Hence in the analysis above $T_i(j, t) = T_i(t)/J$ constant across j. At any location i at time t the state variable $T_i(t)$ and the cost of inputs w_i summarize all we need to know about the distribution of the unit costs across goods there. Moreover, they do so

[3] This literature often sets $J = 1$. For many purposes we can do the same, but for some we cannot, as will become apparent.

[4] See Feller (1968, Chapter VI) as applied in Grossman and Helpman (1991).

through the state variable:

$$\Phi_i(t) = T_i(t)w_i^{-\theta}. \tag{4.7}$$

Hence, in the analysis in Sections 4.1 through 4.3, we can set

$$\Phi = \Phi_i(j,t) = \Phi_i(t)/J,$$

which is common to all goods j. It follows that the order statistics $C^{(k)}$ for unit costs have the same distribution for each good j.

We adopt the convention that, due to the independence of the efficiency draws across j, the probability distribution of the efficiency for any particular good j also describes the distribution of efficiency draws across goods.

>From Section 4.1, the number of ideas that deliver a unit cost less than or equal to c for an individual good is distributed Poisson with parameter $(\Phi_i(t)/J)c^{\theta}$. An immediate implication of our convention is that, across the range of goods, the measure with unit cost less than c is:

$$H_i(c) = \Phi_i(t)c^{\theta}. \tag{4.8}$$

This result will prove useful in applying this framework to monopolistic competition in the next chapter.

We go on to use the probabilistic results from the previous section to make statements that apply across goods. Here we summarize and interpret those results one by one:

1. From Lemma 1, the distribution of the lowest cost $C^{(1)}$ for producing a good is:

$$F_1(c_1) = \Pr[C^{(1)} \leq c_1] = 1 - \exp\left[-\left(\Phi_i(t)/J\right) c_1^{\theta}\right] \tag{4.9}$$

This result gives the distribution of costs delivered by the best, or frontier, ideas. The previous section derived $F_1(c_1)$ as the probability that a particular good j can be produced at a cost less than c_1 using the best technology. Our aggregate assumptions then imply that $JF_1(c_1)$ is the measure of goods that can be produced at cost less than c_1, using best practice. Since $T_i(t)$ reflects how advanced the state of technology is, we can think of $\Phi_i(t) = T_i(t)w_i^{-\theta}$ as translating more advanced technology into lower (on average) unit costs, as tempered by the cost of inputs w_i. The parameter θ reflects the variability of costs, with larger values of θ implying less variability. Under both perfect and Bertrand competition the best ideas are the only ones in use. Moreover, under perfect competition this distribution also corresponds to the distribution of prices, whose moments are given by the next result.

2. From Lemma 2, the moments of $C^{(1)}$ are given by (for $\theta + b > 0$):

$$E\left[\left(C^{(1)}\right)^b\right]^{1/b} = \left[\Gamma\left(\frac{\theta + b}{\theta}\right)\right]^{1/b} \left(\Phi_i(t)/J\right)^{-1/\theta}. \tag{4.10}$$

Under our aggregate assumptions this result yields cross-sectional moments of lowest cost, which are decreasing in $\Phi_i(t)$. A moment that will be of particular interest, requiring a particular value of b, is the CES price index under perfect competition.

3. From Lemma 2, the moments of $C^{(2)}$ are given by (for $2\theta + b > 0$):

$$E\left[\left(C^{(2)}\right)^b\right]^{1/b} = \Gamma\left(\frac{2\theta + b}{\theta}\right)^{1/b} \left(\Phi_i(t)/J\right)^{-1/\theta}. \tag{4.11}$$

Under Bertrand competition, even though only $C^{(1)}$ is in use, $C^{(2)}$ is often the price. Hence this result is useful in constructing the CES price index under Bertrand competition.

4. From Lemma 6, the ratio $M = C^{(2)}/C^{(1)}$ is independent of $C^{(2)}$ and is distributed:

$$F_{2/1}(m) = \Pr\left[M \leq m\right] = 1 - m^{-\theta}. \tag{4.12}$$

Under Bertand competition M is often the markup of price over unit cost. An important consequence of this result for what follows is that markups are unrelated to any features embodied in $\Phi_i(t)$, such as the history of technology or input costs, a feature it shares with the fixed markup of monopolistic competition and quality ladders.

5. From Lemma 3, conditional on $C^{(1)} = c_1$, the distribution of $C^{(2)}$ is:

$$\Pr[C^{(2)} \leq c_2 | C^{(1)} = c_1] = 1 - \exp\left[-\left(\Phi_i(t)/J\right)\left(c_2^\theta - c_1^\theta\right)\right]. \tag{4.13}$$

The lower c_1, the more likely a low $C^{(2)}$. Under Bertrand competition, then, low-cost producers are more likely to charge a lower price, since $C^{(2)}$ is often the price.

6. From Lemma 4, the distribution of the ratio $M = C^{(2)}/C^{(1)}$ given $C^{(1)} = c_1$ is:

$$\Pr\left[M \leq m | C^{(1)} = c_1\right] = 1 - \exp\left[-\left(\Phi_i(t)/J\right) c_1^{\theta}(m^{\theta} - 1)\right]. \tag{4.14}$$

   The lower c_1, the more likely a high markup. Under Bertrand competition, then, low-cost producers are more likely to charge a higher markup.

7. From Lemma 7, any linear homogeneous function of unit costs is homogeneous of degree $-1/\theta$ in $\Phi_i(t)$.

We next turn to how these results can be combined with various assumptions about preferences and market structure to deliver general equilibrium results.

References

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A

4.5 Appendix

This Appendix provides proofs of the Theorem and lemmata stated in the Chapter.

B

4.5.1 Proof of Theorem 1

We first focus on the distribution of the order statistics for techniques with cost less than $\overline{c}$. From Proposition 4.1, the distribution of C given that $C \leq \overline{c}$ is:

$$
\begin{aligned}
F(c|\overline{c}) &= \left(\frac{c}{\overline{c}}\right)^{\theta} \quad c \leq \overline{c} \\
F(c|\overline{c}) &= 1 \qquad\quad\;\; c > \overline{c}
\end{aligned}
$$

The probability a cost is less than c_k is $F(c_k|\overline{c})$ while the probability that it is more than c_{k+1} is $1 - F(c_{k+1}|\overline{c})$. Hence, if there are n techniques with unit cost less than $\overline{c}$, where $c_k \leq c_{k+1} \leq \overline{c}$, the probability that k are less than c_k while the remaining $n-k$ are greater than c_{k+1} is, from the multinomial:

$$
\binom{n}{k} F(c_k|\overline{c})^k [1 - F(c_{k+1}|\overline{c})]^{n-k},
$$

an object closely related to the joint c.d.f. of $C^{(k)}$ and $C^{(k+1)}$. Taking the negative of the cross derivative of this expression with respect to c_k and c_{k+1} gives the joint density of $C^{(k)},\ C^{(k+1)}$:

$$
g_{k,k+1}(c_k, c_{k+1}|\overline{c}, n) = \frac{n!\,[F(c_k|\overline{c})]^{k-1}\,[1 - F(c_{k+1}|\overline{c})]^{n-k-1}\,F'(c_k|\overline{c})F'(c_{k+1}|\overline{c})}{(k-1)!(n-k-1)!},
$$

for $c_{k+1} \geq c_k$ and $n \geq k+1$.[5] For $n < k+1$ we can set $g_{k,k+1}(c_k, c_{k+1}|\overline{c}, n) = 0$. Since, from Proposition 4.1, n is drawn from the Poisson distribution with parameter $\Phi\overline{c}^{\theta}$, the expectation of this joint distribution unconditional on n is:

$$\begin{aligned}
g_{k,k+1}(c_k, c_{k+1}|\overline{c}) &= \sum_{n=0}^{\infty} \frac{\exp(\Phi\overline{c}^{\theta})\left(\Phi\overline{c}^{\theta}\right)^n}{n!} g_{k,k+1}(c_k, c_{k+1}|\overline{c}, n) \\
&= \frac{[F(c_k|\overline{c})]^{k-1}(\Phi\overline{c}^{\theta})^{k+1}\exp[-\Phi\overline{c}^{\theta}F(c_{k+1}|\overline{c})]F'(c_k|\overline{c})F'(c_{k+1}|\overline{c})}{(k-1)!} \\
&\qquad \sum_{n=k+1}^{\infty} \frac{e^{-\Phi\overline{c}^{\theta}[1-F(c_{k+1}|\overline{c})]}\left\{\Phi\overline{c}^{\theta}\left[1-F(c_{k+1}|\overline{c})\right]\right\}^{n-k-1}}{(n-k-1)!} \\
&= \frac{[F(c_k|\overline{c})]^{k-1}(\Phi\overline{c}^{\theta})^{k+1}\exp[-\Phi\overline{c}^{\theta}F(c_{k+1}|\overline{c})]F'(c_k|\overline{c})F'(c_{k+1}|\overline{c})}{(k-1)!} \\
&\qquad \sum_{m=0}^{\infty} \frac{e^{-\Phi\overline{c}^{\theta}[1-F(c_{k+1}|\overline{c})]}\left\{\Phi\overline{c}^{\theta}\left[1-F(c_{k+1}|\overline{c})\right]\right\}^{m}}{m!} \\
&= \frac{[F(c_k|\overline{c})]^{k-1}(\Phi\overline{c}^{\theta})^{k+1}\exp[-\Phi\overline{c}^{\theta}F(c_{k+1}|\overline{c})]F'(c_k|\overline{c})F'(c_{k+1}|\overline{c})}{(k-1)!}
\end{aligned}$$

The last result follows since the summation is over the domain of the Poisson distribution with parameter $\Phi\overline{c}^{\theta}\left[1-F(c_{k+1}|\overline{c})\right]$. Substituting our expression for $F(c|\overline{c})$ we get:

$$g_{k,k+1}(c_k, c_{k+1}|\overline{c}) = \frac{\theta^2}{(k-1)!}\Phi^{k+1}c_k^{\theta k-1}c_{k+1}^{\theta-1}\exp[-\Phi c_{k+1}^{\theta}].$$

[5] See section 4.6 of Hogg and Craig (1995) for generalizations of this result.

By letting $\overline{c} \to \infty$ this joint density is defined for all $c_k > 0$, delivering the joint density of the Theorem. To get the marginal density we calculate:

$$g_k(c_k) = \frac{\theta^2}{(k-1)!}\Phi^{k+1}c_k^{\theta k-1}\int_{c_k}^{\infty} c_{k+1}^{\theta-1}\exp[-\Phi c_{k+1}^{\theta}]dc_{k+1}.$$

4.5.2 Proof of Lemma 1

As is necessary for any cumulative distribution function, $F_k(c_k)$ approaches 1 as $c_k \to \infty$. Furthermore, from Theorem 1, $F_k'(c_k) = g_k(c_k)$ as required.

4.5.3 Proof of Lemma 2

First consider $k = 1$:

$$\begin{aligned} E\left[\left(C^{(1)}\right)^b\right] &= \int_0^{\infty} c^b g_1(c)dc \\ &= \int_0^{\infty} \Phi\theta c^{\theta+b-1}\exp[-\Phi c^{\theta}]dc. \end{aligned}$$

Changing the variable of integration to $v = \Phi c^{\theta}$ and applying the definition of the gamma function, we get:

$$E\left[\left(C^{(1)}\right)^b\right] = \int_0^{\infty}(v/\Phi)^{b/\theta}e^{-v}dv = \left(\Phi^{-1/\theta}\right)^b\Gamma\left[\frac{\theta+b}{\theta}\right],$$

which is well defined for $\theta + b > 0$. In general,

$$\begin{aligned} E\left[\left(C^{(k)}\right)^b\right] &= \int_0^\infty c^b g_k(c)dc \\ &= \frac{\Phi^{k-1}}{(k-1)!}\int_0^\infty c^{b+\theta(k-1)} g_1(c)dc \\ &= \frac{\Phi^{k-1}}{(k-1)!} E\left[\left(C^{(1)}\right)^{b+\theta(k-1)}\right] \end{aligned}$$

Calculating the $b + \theta(k-1)$ moment of $C^{(1)}$ (which can done as long as $b + \theta k > 0$) gives the general result.

B

4.5.4 Proof of Lemma 3

We solve:

$$\begin{aligned} \Pr[C^{(k+1)} &\leq c_{k+1}|C^{(k)} = c_k] = \int_{c_k}^{c_{k+1}} \frac{g_{k,k+1}(c_k, c)}{g_k(c_k)} dc \\ &= \int_{c_k}^{c_{k+1}} \theta\Phi c^{\theta-1} \exp[-\Phi c^\theta + \Phi c_k^\theta] dc, \end{aligned}$$

delivering the result.

B

4.5.5 Proof of Lemma 4

Since

$$\Pr\left[\frac{C^{(k+1)}}{C^{(k)}} \leq m|C^{(k)} = c_k\right] = \Pr\left[C^{(k+1)} \leq mc_k|C^{(k)} = c_k\right]$$

the result follows from Lemma 2.

B

4.5.6 Proof of Lemma 5

We evaluate:

$$\begin{aligned}\Pr[C^{(k)} &\leq c_k|C^{(k+1)} = c_{k+1}] = \int_0^{c_k} \frac{g_{k,k+1}(c, c_{k+1})}{g_{k+1}(c_{k+1})} dc \\ &= \int_0^{k} \theta k \frac{c^{\theta k - 1}}{c_{k+1}^{\theta k}} dc\end{aligned}$$

which upon integrating delivers the result.

B

4.5.7 Proof of Lemma 6

We rewrite:

$$\begin{aligned}\Pr\left[\frac{C^{(k+1)}}{C^{(k)}} \leq m|C^{(k+1)} = c_{k+1}\right] &= \Pr\left[C^{(k)} \geq \frac{c_{k+1}}{m}|C^{(k+1)} = c_{k+1}\right] \\ &= 1 - \Pr[C^{(k)} \leq \frac{c_{k+1}}{m}|C^{(k+1)} = c_{k+1}].\end{aligned}$$

Applying the previous lemma for $c_k = c_{k+1}/m$ delivers the result.