EC8602: Final Exam and Answer Key

December 21, 2004

Answer all the questions.

1. In Klette and Kortum (2004), the state variable for a firm is the integer n, the number of different goods it is currently producing. The profit flow per good is π. Time is continuous. The firm must invest at rate $C(I,n)$ for a Poisson hazard I of moving to state $n+1$. The function C is homogenous of degree 1 in its two arguments; increasing and convex in I. The Bellman equation for the firm is

$$
\begin{aligned}
rV(n) &= \max_I \{\pi n - C(I,n) + I[V(n+1) - V(n)]\} \\
&\quad -\mu n[V(n) - V(n-1)].
\end{aligned}
$$

   (a) Show that for some constant v, the function $V(n) = vn$ satisfies the Bellman equation. **Answer:** Rewrite $C(I,n) = nc(I/n)$ and plug the proposed solution for $V(n)$ into the Bellman equation. The first order condition for the optimal choice of I is $c'(I/n) = v$ so that the solution is of the form $I = \lambda n$ for some constant λ. We now have $rvn = \pi n - nc(\lambda) + \lambda nv - \mu nv$ which implies $v = (\pi - c(\lambda))/(r + \mu - \lambda)$ thus verifying the solution.

   (b) Show that $v \geq \pi/(r+\mu)$. **Answer:** If the firm cannot invest, its value becomes $\pi n/(r+\mu)$. When the firm has the option to invest, its value vn must be at least this large.

   (c) Measuring the size of the firm by n, show (heuristically) that firm growth obeys Gibrat's Law. **Answer:** In the next instant of time dt, a firm of size n will increase in size by 1 with probability $I(n)dt$ and will decrease in size by 1 with probability μndt. Since $I(n) = \lambda n$, the expect change in size is $(\lambda n - \mu n)dt$. Thus the expected proportional growth rate per unit of time is $(\lambda n - \mu n)dt/(ndt) = \lambda - \mu$.

2. In the problem above, the influence of all other firms in the industry is captured by the constant μ. Now consider a problem with a similar setup (continuous time, firm positions moving about as steps on the integers, an interest rate r), but with a dynamic industry structure as in Ericson and Pakes (1995). There are exactly 2 firms in the industry. Let the integers n_1 and n_2 denote the positions of firm 1

and 2, respectively. To keep things simple, suppose a firm is either in position $0, 1$, or 2 and furthermore $n_1 + n_2 = 2$. If firm i is in position $n_i < 2$ (if $n_i = 2$, firm i has no where to go by investing) and invests at rate $C(I_i, n_i)$, it has a Poisson hazard I_i of moving to position $n_i + 1$ (in which case the other firm, i', is knocked down from position $n_{i'}$ to position $n_{i'} - 1$). Denote the profit flow of firm 1 by $\pi(n_1, n_2)$. As in Pakes and McGuire (1994), denote firm 1's value function by $V(n_1, n_2)$.

(a) Write down the Bellman equations for $V(0, 2)$, $V(1, 1)$, and $V(2, 0)$, being explicit about what firm 1 is taking as given. **Answer:** If firm 2 is in state n its innovation hazard is $I(n)$, for $n = 0, 1$ (firm 2 cannot innovate if it is already in state 2). Firm 1 takes these values as given in its value functions:

$$rV(0,2) = \max_I \{\pi(0,2) - C(I,0) + I[V(1,1) - V(0,2)]\}.$$

$$\begin{aligned} rV(1,1) &= \max_I \{\pi(1,1) - C(I,1) + I[V(2,0) - V(1,1)]\} \\ &\quad -I(1)[V(1,1) - V(0,2)]. \end{aligned}$$

$$rV(2,0) = \pi(2,0) - I(0)[V(2,0) - V(1,1)]$$

(b) Write down the definition of an industry equilibrium as in Ericson and Pakes (1995). **Answer:** An industry equilibrium is $V(0, 2)$, $V(1, 1)$, and $V(2, 0)$ given above, where

$$I(0) = \arg\max_I \{-C(I,0) + I[V(1,1) - V(0,2)]\}$$

$$I(1) = \arg\max_I \{-C(I,1) + I[V(2,0) - V(1,1)]\}.$$

(c) Describe as precisely as possible the dynamics of industry structure. **Answer:** Industry structure is given by the number of firms in position $n \in \{1, 2, 3\}$. Thus (since the identity of which firm is in which position is irrelevant) there are just two industry structures: $M = (1, 0, 1)$ and $D = (0, 2, 0)$. The hazard of moving from industry structure M to D is given by $I(0)$, i.e. the innovation rate of the firm in position 0. The hazard of moving from structure D to M is given by $2I(1)$, the combined innovation rate of the two firms when they are each in position 1.

3. Describe carefully how you can introduce exit and entry into the problem above (the simpler the better). Show explicitly how the value functions can be simplified in this case. **Answer:** Suppose that a firm in position 0 has exited. It gets no flow of profit and hence is no different than any other potential entrant. Taken together, potential entrants pay a flow cost $F\eta$ to enter the industry with a Poisson hazard η. If an entrant succeeds, it enters the industry in position 1 (hence the incumbant firm that it faces will always be in position 1 as well). There is an infinite pool of potential entrants so that they cannot expect to make a profit: $F\eta \geq \eta V(1,1)$. Assume $\eta > 0$ so that $V(1,1) = F$. As noted above $V(0,2) = 0$. Since the cost of investment function only applies to size one firms, let $c(I) = C(I,1)$. The value functions simplify to:

$$\begin{aligned} rF &= \max_I \{\pi(1,1) - c(I) + I[V(2,0) - F]\} \\ &\quad -[I(1) + \eta/2]F. \end{aligned}$$

$$rV(2,0) = \pi(2,0) - \eta[V(2,0) - F].$$

These equations simplify further (imposing equilibrium) to

$$V(2,0) = \frac{\pi(2,0)}{r+\eta}$$

$$c'(\lambda) = \frac{\pi(2,0)}{r+\eta} - F$$

$$rF = \pi(1,1) - c(\lambda) + \lambda[V(2,0) - F] - [\lambda + \eta/2]F.$$

The last three equations can be solved jointly for $V(2,0)$, λ, and η.