

A Model of the Firm Size Distribution¹

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1. CONSUMERS

Time is continuous and indexed by t . There is a continuum $He^{\eta t}$ of identical infinitely-lived consumers alive at time t . The population growth rate η is non-negative. Each consumer supplies one unit of labor at every point in time. A typical consumer i has preferences over sequences $\{C_{i,t}\}_{t \geq 0}$ of a composite good defined by the utility function:

$$\mathbb{E} \left[\int_0^\infty e^{-\rho t} \left(\frac{C_{i,t}^{1-\gamma} - 1}{1-\gamma} \right) dt \right]$$

where γ is positive. A continuum of goods of different types make up the composite good. Preferences are additively separable with weights that define the type of a good. The additive separability implies that all goods of the same type and trading at the same price will be consumed at the same rate. Let $c_{i,t}[u, p]$ be consumer i 's consumption at time t of a good of type u that trades at a price p . In equilibrium, there will be a measure U_t of goods that are available at time t , defined on the set of types and prices. The composite good is a version of the one specified in Dixit and Stiglitz (1977). For some $\omega \in (0, 1)$:

$$C_{i,t} = \left[\int u^{1-\omega} c_{i,t}^\omega[u, p] dU_t(u, p) \right]^{1/\omega}$$

The type u of a good can be viewed as measure of quality. Consumer i chooses $c_{i,t}[u, p]$ to minimize the cost of acquiring $C_{i,t}$. This implies:

$$p c_{i,t}[u, p] = P_t (u C_{i,t})^{1-\omega} c_{i,t}^\omega[u, p] \tag{1}$$

where the price index P_t is equal to:

$$P_t = \left[\int u p^{-\frac{\omega}{1-\omega}} dU_t(u, p) \right]^{-\frac{1-\omega}{\omega}}$$

The price elasticity of demand is $-1/(1-\omega)$, and the implied expenditure share for good (u, p) is $u(p/P_t)^{-\omega/(1-\omega)}$.

Consumers own the capital stock of the economy. They can trade in a sequence of complete markets, subject to a present-value borrowing constraint. Since preferences are homothetic and markets are complete, we can assume there is a representative consumer with the same preferences as described here for individual consumers. Aggregate consumption of the composite good is denoted by C_t . The aggregate consumption of a type- (u, p) commodity is written as $c_t[u, p]$.

Along the balanced growth path to be constructed below, per capita consumption will grow at a rate κ defined below. The implied equilibrium interest rate is $r = \rho + \gamma\kappa$.

2. FIRMS

A firm is defined by its unique access to a technology that can be used to produce a particular differentiated good. An existing firm can be continued only at a cost equal to λ_F units of labor per unit of time. A firm of vintage t at age a can use labor $L_{t,a}$ to produce $z_{t,a}L_{t,a}$ units of a differentiated good of type $u_{t,a}$. Given a price $p_{t,a}$, revenues of the firm are given by $R_{t,a} = p_{t,a}z_{t,a}L_{t,a}/P_t$ in units of the composite commodity. The demand function for type- $u_{t,a}$ goods then implies that revenues of the firm are:

$$R_{t,a} = C_{t+a}^{1-\omega} (Z_{t,a}L_{t,a})^\omega \quad (2)$$

where $Z_{t,a} = (u_{t,a}^{1-\omega} z_{t,a}^\omega)^{1/\omega}$ combines the state of preferences and technology. A firm's revenues vary with changes in the type $u_{t,a}$ of its output and its productivity level $z_{t,a}$. The $Z_{t,a}$ are assumed to evolve independently across firms, according to:

$$Z_{t,a} = Z \exp(\zeta t + \zeta_* a + \sigma_* W_{t,a}) \quad (3)$$

where $W_{t,a}$ is a standard Brownian motion. The standard deviation σ_* is taken to be positive. Note that $Z_{t,0} = Ze^{\zeta t}$. Thus ζ is the growth rate at which the productivity of entrants grows over time.

Exogenous factors cause firms to exit at a rate ϵ per unit of time. Firms can also choose to exit when fixed costs are too high to continue. Measured in units of the composite good, the value V_t of a firm entering at time t is given by:

$$V_t = \max_{Q,\tau} \mathbb{E}_t \left[\int_0^\tau e^{-(r+\epsilon)a} (R_{t,a} - w_{t+a} [L_{t,a} + \lambda_F]) da \right] \quad (4)$$

The maximization is subject to (2) and the restriction that choices only depend on the available information.

The aggregate supply of structures and labor grows at a rate η , and every firm requires a fixed input of structures and labor to operate. Along the balanced growth path, the number of firms N_t will grow exactly at the rate η . Firm revenues can be written in terms of the amount of aggregate consumption per firm C_{t+a}/N_{t+a} and $N_{t+a}^{1-\omega} (Z_{t,a}L_{t,a})^\omega$. Thus growth in the number of firms acts like an improvement in productivity. The productivity variable $Z_{t,a}$ grows at an average rate ζ . As a result, the effective growth rate of productivity in the consumption goods sector will be:

$$\kappa = \frac{1}{\omega} [\omega\zeta + (1 - \omega)\eta] \quad (5)$$

along the balanced growth path. Since consumption goods are not perfect substitutes, this growth rate exceeds ζ if the population growth rate is positive. Aggregate consumption at time t will be $C_t = Ce^{(\kappa+\eta)t}$ along the balanced growth path.

2.1 Variable Input Choices

At every point in time, firms choose variable labor $L_{t,a}$ to maximize variable profits $R_{t,a} - w_{t+a}L_{t,a}$, subject to (2). The solution is:

$$\begin{bmatrix} R_{t,a} \\ w_{t+a}L_{t,a} \end{bmatrix} = \begin{bmatrix} 1 \\ \omega \end{bmatrix} \left[\frac{\omega Z_{t,a}}{w_{t+a}} \right]^{\omega/(1-\omega)} C_{t+a}$$

After fixed and variable costs, the resulting flow revenues can be written as:

$$R_{t,a} - w_{t+a}(L_{t,a} + \lambda_F) = w_{t+a}\lambda_F(e^{s_a} - 1)$$

where s_a equals:

$$s_a = S + \frac{\omega}{1-\omega} \left[\ln \left(\frac{Z_{t,a}}{Z_{t,0}} \right) - \zeta a \right] \quad (6)$$

and where S is given by:

$$e^S = \frac{(1-\omega)C}{w\lambda_F} \left(\frac{\omega Z}{w} \right)^{\omega/(1-\omega)} \quad (7)$$

The fact that the factor $e^{s_a} - 1$ does not depend on t relies on (5). The variable s_a measures profitability, and the initial profitability of a new firm is S . If $s_a = 0$, then variable revenues just cover fixed costs, and flow profits are zero. Define the value function $V(\cdot)$ as:

$$V(s) = \max_{\tau} \mathbb{E} \left[\int_0^{\tau} e^{-(r+\epsilon-\kappa)a} (e^{s_a} - 1) da \mid s_0 = s \right]$$

for any s . The profitability of a firm set up at time t follows $ds_a = \mu da + \sigma dW_{t,a}$, where:

$$\begin{bmatrix} \mu \\ \sigma \end{bmatrix} = \frac{\omega}{1-\omega} \begin{bmatrix} \zeta_* - \zeta \\ \sigma_* \end{bmatrix} \quad (8)$$

Profitability has a negative drift when productivity inside the firm is expected to grow more slowly than the productivity of new entrants. Note that the differences in these growth rates and the variance of productivity shocks are magnified significantly when the differentiated goods are close substitutes. The value of a firm entering at time t can now be written as:

$$V_t = w_t\lambda_F V(S)$$

$V(s)$ is the value of a firm relative to its fixed costs when the current state of profitability is s . The value function $V(\cdot)$ only depends on the interest rate and fixed parameters of the economy.

2.2 The Exit Decision

The presence of fixed costs implies a minimum size. Firms with very low productivity choose to exit. The value of a firm must be finite in any equilibrium. The following assumption makes sure that this is the case.

Assumption 1: *The productivity and exogenous exit parameters satisfy:*

$$\rho + \gamma\kappa + \epsilon > \kappa, \rho + \gamma\kappa + \epsilon > \kappa + \mu + \frac{1}{2}\sigma^2$$

where κ is given by (5) and μ and σ are defined in terms of ζ , ζ_* and σ_* by (8).

The first inequality ensures that the fixed costs incurred by the firm have a finite present value even if the firm is never shut down for other than exogenous reasons. The second part implies that the present value of profits before fixed costs is also finite for such a shut-down policy.

The value function $V(s)$ must satisfy the following Bellman equation in the range of s where a firm is not shut down:

$$rV(s) = (\kappa - \epsilon)V(s) + \mathcal{A}V(s) + e^s - 1 \quad (9)$$

where:

$$\mathcal{A}V(s) = \mu DV(s) + \frac{1}{2}\sigma^2 D^2V(s)$$

The return to owning a firm consists of a capital gain $\kappa - \epsilon + \mathcal{A}V(s)/V(s)$ and a dividend yield $(e^s - 1)/V(s)$. It is optimal to shut down a firm when its profitability s falls below some threshold b . Given that the firm is shut down at b , it must be that the value of a firm is zero at that point. This implies the boundary condition $V(b) = 0$. The optimal threshold must be such that V is differentiable at b , and so $DV(b) = 0$. A further boundary condition follows from the fact that the value function cannot exceed the value of a firm that operates without fixed costs. This implies that $V(s) \leq e^s/(r + \epsilon - [\kappa + \mu + \sigma^2/2])$.

With these boundary conditions, the Bellman equation (9) has only one solution. It is given by $V(s) = 0$ for $s \leq b$, and by:

$$V(s) = \frac{1}{r + \epsilon - \kappa} \left[-1 + \left(\frac{\xi}{1 + \xi} \right) e^{s-b} + \left(\frac{1}{1 + \xi} \right) e^{-\xi(s-b)} \right] \quad (10)$$

for all $s \geq b$. The exit barrier b satisfies:

$$e^b = \left(\frac{\xi}{1 + \xi} \right) \frac{r + \epsilon - [\kappa + \mu + \frac{1}{2}\sigma^2]}{r + \epsilon - \kappa} \quad (11)$$

and the coefficient ξ is given by $[\mu + \sqrt{\mu^2 + 2\sigma^2(r + \epsilon - \kappa)}]/\sigma^2$. Note that $r + \epsilon > \kappa$ implies $\xi > 0$. It follows that $V(s)$ is strictly increasing and convex on (b, ∞) , with an asymptote equal to the present value of $\{e^{sa} - 1\}_{a \geq 0}$.

2.3 Entry

One entry per unit of time costs λ_E units of labor. Along the balanced growth path, entry takes place at all times. The resulting zero-profit condition is:

$$\lambda_E = \lambda_F V(S) \tag{12}$$

This zero-profit condition depends implicitly on the interest rate r , via $V(\cdot)$, and on steady-state wages and aggregate consumption, via S .

The returns to entry can be made arbitrarily small or large by taking $C/w^{1/(1-\omega)}$ to be small or large, respectively. Thus (12) implies a unique equilibrium value for $C/w^{1/(1-\omega)}$. In turn this pins down the equilibrium value of S that determines profitability upon entry. It is not difficult to see that e^{S-b} is increasing in λ_E . In equilibrium, the more difficult it is to enter, the higher must be the initial profitability of firms that enter.

The maintained assumption here is that new entrants cannot copy the productivity parameter $Z_{t,a}$ of an incumbent. One can allow there to be a market in which incumbents can licence their technology to potential entrants, using a contract that completely specifies the actions that the entrant can undertake. Such licencees would then be able to operate the same linear technology as the incumbent licensor. Assuming that any potential licensee also faces a fixed cost, there are no incentives to set up such new firms. In this environment, it is optimal to incur the fixed cost of running a firm only once.

3. THE STATIONARY DISTRIBUTION OF FIRM CHARACTERISTICS

Every final goods firm needs a fixed amount of labor to produce. The aggregate labor supply grows at a non-negative rate η . Along the balanced growth path to be constructed below, the number of firms also grows at the rate η . The following assumption is needed to guarantee that the stationary distribution of firm characteristics is such that average profitability is finite.

Assumption 2: *The productivity parameters satisfy:*

$$\epsilon + \eta > \mu + \frac{1}{2}\sigma^2$$

where μ and σ are defined in (8).

Note that $\mu + \sigma^2/2$ is the drift of the profitability measure e^{s_a} . Thus Assumption 2 requires that the profitability of a typical incumbent firm is not expected to grow faster than the sum of the population growth rate and the exogenous exit rate. The growth rate of profitability among surviving firms will be greater than $\mu + \sigma^2/2$. If $\epsilon + \eta$ is zero then μ must be negative, but it can be positive otherwise.

3.1 Age and Profitability

Along the balanced growth path, there will be a measure of firms growing at a rate η , and defined on the set of possible ages and profitability levels. The density of this measure at date t can be written as $m(a, s)Ie^{\eta t}$, where $Ie^{\eta t}$ is the number of new firms entering per unit of time. The density m must satisfy a version of the Kolmogorov forward equation:

$$D_a m(a, s) = -(\epsilon + \eta)m(a, s) - \mu D_s m(a, s) + \frac{1}{2}\sigma^2 D_{ss} m(a, s) \quad (13)$$

for all $a > 0$ and $s > b$. The first term on the right-hand side of (13) reflects the exogenous exit of firms and the fact that the measure of firms grows over time. The remaining two terms describe how the density $m(a, s)$ evolves as a result of changes in the profitability of individual firms.

The boundary value $m(0, s)$ is a unit point mass at S :

$$\int_0^s m(0, x)dx = \begin{cases} 1 & s \geq S \\ 0 & \text{otherwise} \end{cases} \quad (14)$$

A further boundary condition is given by the requirement that $m(a, b) = 0$ for all $a > 0$. This condition arises from the fact that firms exit at b while none enter starting from profitability levels below b . To ensure that the measure of firms is finite, it must also be the case that $m(a, s)$ goes to zero for large (a, s) . The solution of (13)-(14) is given by:

$$m(a, s) = e^{-(\epsilon+\eta)a}\psi(a, s|S) \quad (15)$$

for all $a > 0$ and all $s > b$, where:¹

$$\psi(a, s|S) = \frac{1}{\sigma\sqrt{a}} \left[\phi\left(\frac{s - S - \mu a}{\sigma\sqrt{a}}\right) - e^{-\frac{2\mu}{\sigma^2}(s-b)} \phi\left(\frac{s + S - 2b - \mu a}{\sigma\sqrt{a}}\right) \right]$$

¹The standard normal density and distribution functions are denoted by ϕ and Φ , respectively.

By differentiating, one can check that the two terms that define $e^{-(\epsilon+\eta)a}\psi(a, s|S)$ both satisfy (13). For small values of a , the first term approximates a normal probability density that puts almost all probability close to $s = S$. The second term converges to zero as a goes to zero, since $s + S > 2b$. As a result, $m(a, s)$ converges to the value required by the boundary condition (14), as a goes to zero. The fact that $\psi(a, s|b) = 0$ ensures that the boundary condition $m(a, b) = 0$ holds.

3.2 Profitability—The Power Law

Define the parameters θ and θ_* as follows:

$$\theta = \frac{1}{\sigma^2} \left(-\mu + \sqrt{\mu^2 + 2\sigma^2[\epsilon + \eta]} \right), \theta_* = \frac{1}{\sigma^2} \left(\mu + \sqrt{\mu^2 + 2\sigma^2[\epsilon + \eta]} \right)$$

Assumption 2 implies that $\theta > 1$ and the fact that $\epsilon + \eta$ is non-negative implies that θ_* is non-negative as well. If $\epsilon + \eta = 0$, then θ is simply equal to $-2\mu/\sigma^2$ and $\theta_* = 0$. By integrating $e^{-(\epsilon+\eta)a}\psi(a, s|S)$ one can verify that the density of age and profitability is:

$$\pi(a, s|S) = \left(\frac{1 - e^{-\theta_*(S-b)}}{\epsilon + \eta} \right)^{-1} e^{-(\epsilon+\eta)a}\psi(a, s|S) \quad (16)$$

for all $s \geq b$. The implied density of profitability given an initial profitability $S > b$ is:²

$$\pi(s|S) = \frac{\theta}{\theta + \theta_*} \frac{\min \left\{ e^{[\theta+\theta_*(s-b)]}, e^{[\theta+\theta_*(S-b)]} \right\} - 1}{e^{\theta(s-b)} \left[\frac{e^{\theta_*(S-b)} - 1}{\theta_*} \right]} \quad (17)$$

for all $s \geq b$. The kink at $s = S$ is a result of the entry that takes place at S . Conditional on $s \geq S$, the density of e^s implied by (17) is a Pareto density with tail probabilities of the form $e^{-\theta s}$. This is the power law found in firm size data. The parameter θ will be referred to as the “tail index” of the profitability distribution.

The size of a firm measured by revenues or variable inputs is proportional to a factor depending on fixed characteristics, and profitability. The model described here can therefore account for a power law in firm size data if profitability is the main determinant of size. Unlike the Pareto density, and like the log-normal density often considered as an alternative, the density (17) is upward sloping for low size levels. Unlike the log-normal, the support of (17) is bounded below, as a result of the fixed costs that cause low-productivity firms to exit.

²Alternatively, note that by integrating out a from (13)-(14) one obtains an inhomogeneous second-order differential equation in s with a characteristic equation $\epsilon + \eta + \mu z - \sigma^2 z^2/2 = 0$. The roots of this equation are θ and θ_* . The differential equation is solved by (17).

The mean of e^{s-b} is:

$$\int_b^\infty e^{s-b} \pi(s|S) ds = \frac{\theta}{\theta - 1} \left[\frac{\frac{1 - e^{-(1+\theta_*)(S-b)}}{1+\theta_*}}{\frac{1 - e^{-\theta_*(S-b)}}{\theta_*}} \right] e^{S-b} \quad (18)$$

Given that $\epsilon + \eta$ is non-negative, Assumption 2 is exactly equivalent to $\theta > 1$. Thus an approximate version of Zipf's law can hold in this economy, but it cannot hold exactly. The stationary density (17) would be well defined even for $\theta > 0$, but without $\theta > 1$ the mean of e^{s-b} would not be finite and aggregate quantities would not be well defined. The average profitability in a large cross-section of firms would not converge as the number of firms grows without bound. The abstraction of an economy with balanced growth and a continuum of infinitesimal firms is not well defined in that case.^{3 4}

The right-hand side of (18) is greater than $\theta/(\theta-1)$. Thus the ratio of the profitability of the average firm relative to an exiting firm is bounded below by $\theta/(\theta-1)$. When the productivity growth rate of existing firms is close to that of new entrants, θ will be close to 1 and the lower bound $\theta/(\theta-1)$ will be a very large number. In such an environment, even the average firm is much larger and more profitable than the firms that exit. If entry is easy, then $S-b$ will be small. In that case, entering firms will also be small relative to the average firm, and the power law will apply over much of the range of firm sizes.

To emphasize the importance of random productivity growth in shaping the distribution of firm characteristics, it is instructive to consider what happens as the variance of productivity shocks, σ^2 , goes to zero. For simplicity, suppose that $\epsilon + \eta = 0$. Assumption 2 then requires $\mu < 0$ and at $\sigma^2 = 0$ one obtains $\xi = (r + \epsilon - [\kappa + \lambda]) / |\mu|$ and $b = 0$. Firms exit immediately when they no longer break even since there is no option value to continuing operations. An entering firm starts with profitability s_0 , and this profitability will then decline linearly to 0, at which point the firm exits. As σ^2 goes to 0, the tail-index θ grows without bound. Using (17) one can verify that the stationary profitability distribution $\pi(\cdot|S)$ converges to a uniform distribution on $(0, S)$. In this

³Gabaix (2003) has proposed to move away from this abstraction and to consider an economy with large firms to generate aggregate shocks.

⁴Suppose the exogenous exit and population growth rates are zero. Consider the limiting distribution obtained by letting s_0 go to b . This turns the profitability process of a dynasty of firms into a Brownian motion with a negative drift and a reflecting barrier at b . The resulting distribution for e^s is a Pareto distribution on $e^s \geq e^b$ with mean $e^b \theta / (\theta - 1)$. In Gabaix (1999), e^s is the size of a city relative to the average city size. This must have mean 1, and so $\theta = 1 / (1 - e^b)$. The explanation given in Gabaix (1999) for Zipf's law for relative city sizes is that b must be very small.

limiting economy, the largest and most profitable firm conditional on fixed characteristics is the most recent entrant. This is in sharp contrast to what is found in the data. The randomness in productivity growth generates a selection mechanism by which the typical firm can be much larger and productive than recent entrants.

4. THE BALANCED GROWTH PATH

The number of firms grows at a rate η and per capita consumption and wages grow at the rate κ determined in (5). The zero-profit condition (12) determines S and thus the density m of firm profitability, per entering firm. In units of labor, aggregate output and variable labor are:

$$\begin{bmatrix} Y/w \\ L \end{bmatrix} = I \begin{bmatrix} 1 \\ \omega \end{bmatrix} \frac{\lambda_F}{1-\omega} \int_b^\infty e^s m(s) ds \quad (19)$$

The aggregate amount of labor required for setting up and operating firms is:

$$\begin{bmatrix} L_E \\ L_F \end{bmatrix} = I \begin{bmatrix} \lambda_E \\ \lambda_F \int_b^\infty m(s) ds \end{bmatrix} \quad (20)$$

Market clearing conditions now determine entry and wages. The labor market clearing condition $H = L + L_E + L_F$ determines I :

$$H = I \left(\lambda_E + \lambda_F \left[\int_b^\infty m(s) ds + \frac{\omega}{1-\omega} \int_b^\infty e^s m(s) ds \right] \right) \quad (21)$$

The goods market clearing condition $C = Y$ together with (19) and the definition of S in (7) determine the wage. This yields:

$$w = \omega Z \left[I \int_b^\infty e^{s-S} m(s) ds \right]^{(1-\omega)/\omega} \quad (22)$$

This establishes the following proposition.

Proposition 1: *If Assumptions 1-2 hold, then there exists a balanced growth path.*

A proportional reduction in the entry and fixed costs (λ_E, λ_F) does not affect the zero profit condition. The value of S and the size distribution $m(\cdot)$ therefore do not change. It follows from (21) that I increases in proportion to the decline in entry and fixed costs. In turn, this increases the level of wages with an elasticity $(1-\omega)/\omega$, by (22). The definition of S implies that $C/(\lambda_F w^{1/(1-\omega)})$ does not change. Consumption therefore increases with an elasticity $(1-\omega)/\omega$.

5. TECHNOLOGY SPILLOVERS

Firms described here are engaged in learning-by-doing. By paying a fixed cost, firms can continue to produce and generate productivity improvements. So far, the productivity of potential entrants has been taken to be exogenous. The following makes this initial productivity endogenous by assuming that new entrants can copy from incumbents. Incumbents can also copy from other incumbents, but only by first exiting.

Potential entrants would like to adopt the best technology employed by incumbent firms, but they cannot. In the modified economy, let M_t be the cross-sectional distribution of productivity at time t , and suppose that the initial productivity of a new entrant at time t is given by:

$$Z_{t,0} = \varphi \left[\int z^{\omega/(1-\omega)} dM_t(z) \right]^{(1-\omega)/\omega} \quad (23)$$

for some positive spillover coefficient φ . Recall that the cross-sectional distribution of firm size and profitability at time t is determined by the distribution of $Z_{t-a,a}^{\omega/(1-\omega)}$. Thus (23) says that potential entrants can adopt a technology with a productivity that is equal to some fraction or multiple φ of the productivity of firms of average size and profitability. Following entry at time t , the productivity of a firm at time $t + a$ is:

$$Z_{t,a} = Z_{t,0} \exp(\zeta_* a + \sigma_* W_{t,a})$$

as before. The specification (23) makes the technology available to entrants depend on the distribution of technologies used by incumbents. In turn, this distribution depends on the equilibrium amount of entry and on the exit decisions of incumbents.

To construct a balanced growth path, conjecture that $Z_{t,0} = Z e^{\zeta t}$ for some Z and ζ to be determined. If ζ is large enough, then this implies a stationary firm size distribution as described before. The resulting productivities at time t are given by:

$$Z_{t-a,a} = Z e^{\zeta t + (s_a - S)(1-\omega)/\omega} \quad (24)$$

and s_a is distributed according to $\pi(\cdot|S)$. The spillover (23) therefore implies:

$$1 = \varphi \int_b^\infty e^{s-S} \pi(s|S) ds \quad (25)$$

This adds an equilibrium condition to the ones obtained earlier for an economy in which technological progress is exogenous. The right-hand side of (25) depends on ζ via b , S and $\pi(\cdot|S)$. The equilibrium conditions for these variables are as before. Observe that none of these equilibrium conditions depend on Z . Thus (25) together with the

equilibrium conditions for b , S and $\pi(\cdot|S)$ determine the equilibrium growth rate ζ . The following proposition is shown in the appendix.

Proposition 2: *If the subjective discount rate ρ is large enough, then the economy has a balanced growth path for any spillover parameter $\varphi > 0$. The size distribution converges to Zipf's Law as φ goes to zero. The average level of firm productivity depends on initial conditions.*

By construction, (25) ensures that the average size and profitability of firms is finite. Thus the equilibrium value of ζ must be such that Assumption 2 is satisfied. Assumption 1 can be made to hold by taking ρ large enough. As φ goes to zero, (25) implies that the average size of firms grows without bound. By (18), θ must go to one, and so Zipf's Law holds in the limit.

Suppose the economy starts at date 0 with an initial productivity distribution $M_0(z)$ such that the distribution of $s = S + [\omega/(1 - \omega)] \ln(z/Z)$ has a density $\pi(s|S)$. Then Z is the average productivity at time 0 and the productivity distribution at time t will satisfy:

$$M_t(z e^{-\zeta t}) = M_0(z)$$

along the balanced growth path. The balanced growth path for wages and consumption then follows from (7) and (22). In this economy, initial conditions matter. The initial distribution of productivity determines the level of the path along which the economy will grow.

When $\epsilon + \eta = 0$ and $\gamma = 1$, one can show that the growth rate of the economy is a monotone increasing function of φ . A strong spillover discourages technological progress among incumbents. The rate at which incumbents exit —and thereby destroy the information generated by past learning-by-doing— is increasing in the strength of the spillover.⁵ But the firms that exit are the ones that have not been very successful, and entrants can start anew with a better technology. The result is rapid technological progress and a size distribution with a thin tail. The empirical size distribution has a fat tail, and so the spillover parameter must be small.

⁵The model has no intensive margin.